Algorithms on Real Parametric Curves: topology and other geometric features

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Biographie – I am currently in the second year of my PhD, at Sorbonne Université (Paris 6), in the laboratory IMJ-PRG, under the advisement of Fabrice Rouillier and Elias Tsigaridas. Also a member of OURAGAN team at INRIA Paris. My PhD is supported by the FSMP's MathInParis PhD fellowship program. My research focuses on computational nonlinear algebra and computational geometry.

Resumé :

Parametric curves constitute a classical and important topic in computational algebra and geometry [15] that constantly receives attention, e.g., [14, 6, 4, 16]. The motivation behind the continuous interest in efficient algorithms for computing with parametric curves emanates, among others reasons, by the omnipresence of parametric representations in computer modeling and computer aided geometric design, e.g., [8].

The first part of our contribution [11] focuses on computing the *topology of a real parametric curve*, that is, the computation of an abstract graph that is isotopic [3, p. 184] to the curve in the embedding space. We design a complete algorithm, `PTOPO`, that applies directly to parametric curves of any dimension. We consider different characteristics of the parametrization, like properness and normality, before computing the singularities and other interesting points on the curve. These points are necessary for representing the geometry of the curve, as well as for producing a certified visualization of plane and space curves.

Our method exploits the benefits of the parametric representation and does not resort to implicitization. When the parametrization involves polynomials of degree at most d and maximum bitsize of coefficients τ, then the worst case bit complexity of `PTOPO` is $\widetilde{\mathcal{O}}_B(nd^6 + nd^5\tau + d^4(n^2 + n\tau) + d^3(n^2\tau + n^3) + n^3d^2\tau)$. This bound matches the current record bound $\widetilde{\mathcal{O}}_B(d^6 + d^5\tau)$ for the problem of computing the topology of a plane algebraic curve given in implicit form. For plane and space curves, if $N = \max\{d, \tau\}$, the complexity of `PTOPO` becomes $\widetilde{\mathcal{O}}_B(N^6)$, which improves the state-of-the-art result, due to Alcázar and Díaz-Toca [1], by a factor of N^{10}. In the same time complexity, we obtain a graph whose straight-line embedding is isotopic to the curve. However, visualizing the curve on top of the abstract graph construction, increases the bound to $\widetilde{\mathcal{O}}_B(N^7)$. For curves of general dimension, we can also distinguish between ordinary and non-ordinary real singularities and determine their multiplicities in the same expected complexity of `PTOPO` by employing the algorithm of Blasco and Pérez-Díaz [2]. We have implemented `PTOPO` in MAPLE for the case of plane and space curves [1]. Our experiments illustrate its practical nature.

At the second part of our contribution[2], we study the *convex hull of parametric curves in* $\mathbb{R}^2$. In general, convex hull computation is one of the fundamental problems of computational geometry; given a set of geometric objects in $\mathbb{R}^d$, one is interested in the minimal convex set that includes all of them. Our motivation to study this particular problem, stems from various applications in control theory [13] and in machine learning [17]. Convex hulls of parametric curves also appear in chemical engineering as indicated by [5] and references therein.

[1] `https://webusers.imj-prg.fr/~christina.katsamaki/ptopo/`
[2] to be available online soon

We design an exact algorithm that computes a boundary description of the convex hull of a parametric curve in $\mathbb{R}^2$. The parametric curve is defined over a real interval, such that the image is a compact subset of $\mathbb{R}^2$. The boundary of the convex hull consists of a combination of arcs of the curve and of line segments connecting points on it. We follow closely the algorithm presented in [12]; this algorithm was designed for plane curves given in implicit form by an irreducible polynomial but we adapt it accordingly to the parametric case. The main part of our contribution is the complexity result; our algorithm runs in $\widetilde{\mathcal{O}}_B(d^7\tau)$, where again d is the maximum degree of the polynomials involved in the parametrization and τ is the maximum bitsize of coefficients. Computations involve real root isolation of 3×3 polynomial systems with additional properties, such as being symmetric with respect to a group of variables or being diagonal, which we exploit to arrive to the final complexity. The representation of the boundary of the convex hull obtained at the output of our algorithm is susceptible to numerical computations to approximate the area of the convex hull up to an arbitrary precision.

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