Global entropy stability for a class of unlimited high-order schemes for hyperbolic systems of conservation laws

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Biographie – I am graduated from Ecole Centrale Paris and University of Nantes. In 2020, I decided to pursue my academic background with a thesis. My thesis is funded by a *contrat doctoral handicap.*

Resumé : We consider a hyperbolic system of conservation law

$$\partial_t w + \partial_x f(w) = 0, \quad x \in \mathbb{R},\, t > 0, \tag{1}$$

where f is a smooth given function whose its jacobian is diagonalizable in $\mathbb{R}$. According to [3], it is well known that the solution of the system (1) may develop discontinuities and the weak solutions are not unique. In order to ensure the uniqueness of the solution, the initial system (1) has to be endowed with entropy inequalities

$$\partial_t \eta(w) + \partial_x G(w) \leq 0, \tag{2}$$

where η is a given convex function and G is an entropy flux function verifying ${}^t\nabla\eta(w)\, {}^t\nabla f(w) = {}^t\nabla G(w)$. Integrating in space the entropy inequality (2) leads to a global entropy stability

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \eta(w(x,t))\, \mathrm{d}x \leq 0. \tag{3}$$

From a numerical point of view, the solutions of (1) are approximated on a grid $\left(x_{i+1/2}\right)_{i\in\mathbb{Z}}$ with a finite volume scheme writes

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} + \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{\Delta x} = 0, \tag{4}$$

where w_i^n is the average of w on the cell i at time t^n and $\mathcal{F}_{i+1/2}$ denotes a numerical flux. At a discrete level, the inequality (3) reads

$$\sum_{i\in\mathbb{Z}} \eta\left(w_i^{n+1}\right) \Delta x \leq \sum_{i\in\mathbb{Z}} \eta\left(w_i^n\right) \Delta x. \tag{5}$$

The design of an accurate numerical scheme (4) satisfying the discrete inequality (5) is very challenging and often requires additional limitations techniques [6, 4, 2].

In this context, we introduce a new class of high order free of limitation numerical schemes designed from a perturbation of the standard HLL solver [1]. Thanks to a Taylor expansion in the perturbation, we derive a high order approximation of the flux at the interface. Mixing the discretizations at the cell and at the interface, we prove the global discrete entropy stability property.

The accuracy and stability performances of these schemes will be illustrated in the cases of Burger and Euler equations. For instance, Figure 1 shows the results of the second and fourth order schemes with the Euler equations and for the Sod test case [5]. Some extensions in 2D on unstructured meshes will be eventually presented.

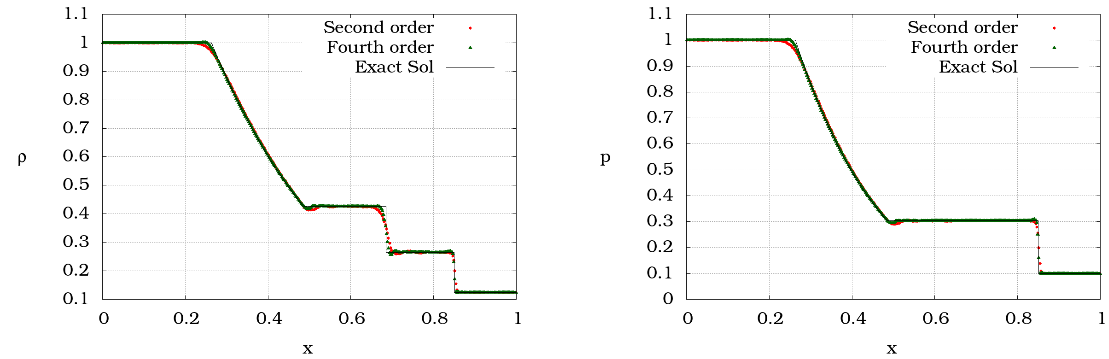

Figure 1: Density and pressure results for the second and the fourth order schemes with the Euler equations and for the Sod test case [5].

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