Relaxation limit for the aggregation equation with pointy potential

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Biographie – Je suis en thèse à l'université de Lyon 1 sous la direction de Frédéric Lagoutière sur des sujets portant sur l'équation d'agrégation et équations de type Keller-Segel, et ayant diverses applications notamment en biologie. Concrètement, je travaille surtout en EDP et analyse numérique. Ma thèse est financée par l'ED Infomaths de Lyon.

Resumé : The so-called aggregation equation is used among the applied mathematics community to model the dynamics of a population of individuals in interaction (chemotaxis, crowd motion). Given an interaction potential $W : \mathbb{R}^d \to \mathbb{R}$ and a initial density of population ρ_0, that we assume to be a probability measure, the aggregation problem is the following nonlocal and nonlinear evolution PDE:

$$\partial_t \rho + \nabla \cdot (a[\rho]\rho) = 0, \tag{1a}$$

$$a[\rho](x) := -\nabla W * \rho(x) = -\int_{y\in\mathbb{R}^d} \nabla W(x-y)\ \mathrm{d}\rho(y), \tag{1b}$$

$$\rho_{|t=0} = \rho_0. \tag{1c}$$

The two phenomena in competition in this model are the concentration of population and the transport phenomenon. When the potential W is pointy, that is, essentially, Lipschitz continuous and $\lambda-$convex but with a Lipschitz singularity at the origin, it is known that weak solutions may blow up in finite time [1, 2] in any L^p norm for $p > 1$. The well-posedness of equation (1) for weak measure-valued solutions has then been proved in [3, 4].

In this work with Benoît Fabrèges, Frédéric Lagoutière and Nicolas Vauchelet, we consider a relaxation approximation of (1) in one space dimension, in the spirit of Jin-Xin [7]. The point is to obtain convergence results in both the continuous and the discrete setting (we propose some suitable numerical schemes) as the relaxation parameter goes to 0, along with convergence rates.

For a given $c > ||W||_{Lip}$, the relaxation system reads:

$$\partial_t \rho^\varepsilon + \partial_x \sigma^\varepsilon = 0, \tag{2a}$$

$$\partial_t \sigma^\varepsilon + c^2 \partial_x \rho^\varepsilon = \frac{1}{\varepsilon}(a[\rho^\varepsilon]\rho^\varepsilon - \sigma^\varepsilon) \tag{2b}$$

$$a[\rho^\varepsilon] = -W' * \rho^\varepsilon, \tag{2c}$$

with initial data ρ_0 and $\sigma_0 := a[\rho_0]\rho_0$.

In this talk, we will see that the formal convergence of ρ^ε towards the unique solution ρ of (1) can be made rigorous in the sense of measures, for arbitrary pointy potentials.

We will then focus on the special case $W(x) = |x|$, in which a correspondance between (1) and a Burgers-type conservation law holds, and allows us to obtain convergence estimates along the lines of Kastoulakis-Tzavaras [8].

In order to illustrate this convergence result, we also provide numerical schemes for the relaxation system (2) that are asymptotic preserving (AP), that is, they degenerate towards convergent schemes for (1) when the relaxation parameter ε goes to 0.

In a first approach, we propose a splitting scheme where we split the transport part and the source term in system (2). The resulting scheme is simple to implement and satisfies the AP property, as it converges when $\varepsilon \to 0$ to a Rusanov-type scheme. The second approach relies on a well-balanced discretization in the spirit of [5, 6]. This scheme is more expensive to implement than the first scheme, but presents less numerical diffusion, as it is illustrated by our numerical results. We also present numerical tests for the convergence rates.

The talk will be given in French if it suits everyone.

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